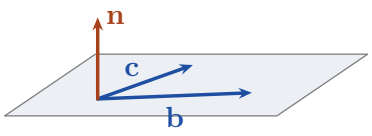
Vectors Practice - Method Guide

Method 1 Find the Cartesian equation of a plane

Use this to turn the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ into $ax + by + cz = k$, or to find a plane from three points.

Example Find the Cartesian equation of this plane.

$$\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\2\\1\end{pmatrix}$$



The method	In the example
Step 1. Pick out a point and two directions. The first vector is a point on the plane. The vectors after λ and μ are directions that lie in the plane.	Point on the plane: $(1, 0, 2)$ Directions in the plane: $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix}$
Step 2. Make an unknown normal perpendicular to both directions. The normal $\mathbf{n}$ sticks straight out of the plane, so it is at right angles to both directions. "At right angles" means the scalar product is zero.	Let the normal be $\mathbf{n}$, with components a, b, c. $\begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = 0$ gives $a + b = 0$ $\begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}0\\2\\1\end{pmatrix} = 0$ gives $2b + c = 0$
Step 3. Solve the two equations. Two equations cannot fix three unknowns. Write two of them in terms of the third, then choose any non-zero value for it.	From $a + b = 0$: $a = -b$ From $2b + c = 0$: $c = -2b$ Choosing $b = 1$: $a = -1$ $c = -2$ so $\mathbf{n} = \begin{pmatrix}-1\\1\\-2\end{pmatrix}$
Check *The normal's scalar product with each direction should be zero.*	$\mathbf{n} \cdot \mathbf{b} = -1 + 1 + 0 = 0$ $\mathbf{n} \cdot \mathbf{c} = 0 + 2 - 2 = 0$
Step 4. Use the normal's numbers as the coefficients of x, y, z. Only the number k on the right is still unknown.	$-1x + 1y - 2z = k$
Step 5. Substitute the point to find k. Any point on the plane must fit the equation.	Using the point $(1, 0, 2)$: $k = -1(1) + 1(0) - 2(2) = -5$ so $-x + y - 2z = -5$ Multiplying through by -1 to tidy up: $x - y + 2z = 5$

Given three points A, B, C: use $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as the directions, then start at Step 2.

Given a line and a point C: use the line's direction and $\overrightarrow{AC}$, where A is any point on the line.

Method 2 Find where a line meets a plane

Use this to find the point where a line crosses a plane.

Example Find where this line meets this plane.

$$\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\1\end{pmatrix} \qquad x + 2y + 3z = 14$$

The method	In the example
Step 1. Write x, y and z for a general point on the line. Each value of λ gives one point on the line. Read across each row of the line's equation.	$x = 1 + 2\lambda \qquad y = 2 - \lambda \qquad z = \lambda$
Step 2. Substitute these into the plane's equation. A point is on the plane when its coordinates fit the plane's equation. So we look for the value of λ that makes this happen.	$x + 2y + 3z = 14$ $\downarrow$ replace x, y, z $(1 + 2\lambda) + 2(2 - \lambda) + 3(\lambda) = 14$
Step 3. Solve for λ. Expand the brackets and collect terms.	$1 + 2\lambda + 4 - 2\lambda + 3\lambda = 14$ $5 + 3\lambda = 14$ $\lambda = 3$
Step 4. Substitute λ into x, y, z. This gives the coordinates of the point where the line meets the plane.	$x = 1 + 2(3) = 7 \qquad y = 2 - 3 = -1 \qquad z = 3$ $P(7, -1, 3)$
Check *The point should fit the plane's equation.*	$7 + 2(-1) + 3(3) = 7 - 2 + 9 = 14$

If the line is given as $x - 1 = \dfrac{y+3}{5} = \dfrac{z}{4}$: all three expressions are equal to the same number. Call it t, and rearrange each to get $x = 1 + t$, $y = -3 + 5t$, $z = 4t$. Then start at Step 2.

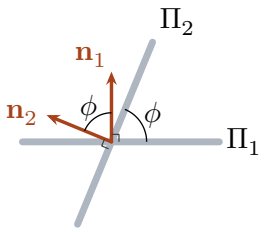
If λ cancels out in Step 3: the line is parallel to the plane. If you are left with something false, such as $-1 = 5$, the line never meets the plane. If it is true, the whole line lies in the plane.

Method 3 Find the angle between two planes (or two lines)

Use this to find the acute angle between two planes, or between two lines.

Example Find the acute angle between these planes.

$2x + 2y - z = 3$ $\qquad$ $3x - 6y + 2z = 7$



The method

In the example

Step 1. Write down the normal of each plane.

The normal's numbers are the coefficients of x, y, z. The angle between two planes is the same as the angle between their normals (the diagram shows two planes edge-on).

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$$

Step 2. Find the scalar product of the normals.

Multiply matching components, then add.

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (2)(3) + (2)(-6) + (-1)(2)$$
$$= 6 - 12 - 2 = -8$$

Step 3. Find the length of each normal.

Square each component, add, and square root.

$$|\mathbf{n}_1| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$$
$$|\mathbf{n}_2| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{49} = 7$$

Step 4. Use the scalar product formula.

The modulus signs make the top positive, which gives the acute angle.

$$\cos\phi = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{|-8|}{3 \times 7} = \frac{8}{21}$$
$$\phi = \cos^{-1}\left(\tfrac{8}{21}\right) \qquad \phi = 67.6°$$

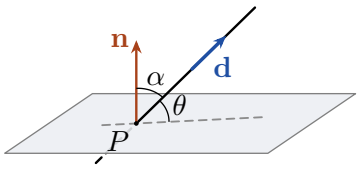
For two lines: do exactly the same, using the two direction vectors in place of the normals.

Method 4 Find the angle between a line and a plane

Use this to find the acute angle between a line and a plane.

Example Find the acute angle between this line and plane.

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad 3x + 4z = 1$$



The method

In the example

Step 1. Write down the line's direction and the plane's normal.

The direction is the vector after λ. The normal's numbers are the coefficients of x, y, z (here there is no y term, so its coefficient is 0).

$$\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

Step 2. Find their scalar product.

Multiply matching components, then add.

$$\mathbf{d} \cdot \mathbf{n} = (2)(3) + (1)(0) + (-2)(4)$$
$$= 6 + 0 - 8 = -2$$

Step 3. Find the length of each vector.

Square each component, add, and square root.

$$|\mathbf{d}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$
$$|\mathbf{n}| = \sqrt{3^2 + 0^2 + 4^2} = \sqrt{25} = 5$$

Step 4. Find the angle between the line and the normal.

The scalar product formula gives the angle α between the two vectors used: the line's direction and the plane's normal. The modulus signs make it the acute angle.

$$\cos\alpha = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}|\,|\mathbf{n}|} = \frac{|-2|}{3 \times 5} = \frac{2}{15}$$
$$\alpha = \cos^{-1}\left(\tfrac{2}{15}\right) = 82.3^\circ$$

Step 5. Subtract from 90°.

The normal is at right angles to the plane. So the angle θ between the line and the plane is what is left of the right angle.

$$\theta = 90^\circ - 82.3^\circ$$
$$\theta = 7.7^\circ$$

What this angle means: θ is the angle between the line and its shadow on the plane (the dashed line, see Method 8). It is the smallest angle the line makes with any line in the plane through P.

A shortcut: because $\theta = 90^\circ - \alpha$, we have $\sin\theta = \cos\alpha$. So Steps 4 and 5 can be done in one go: $\sin\theta = \frac{2}{15}$ gives $\theta = 7.7^\circ$ directly.

Method 5 Find the distance from a point to a plane

Use this to find the closest point on a plane to a given point, and how far away it is.

Example Find the point F on the plane $x + 2y + 2z = 1$ closest to $Q(4, 5, 7)$, and find QF.

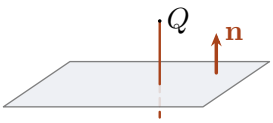
The method

In the example

Step 1. Write the line through Q in the direction of the normal.

The shortest way to a plane is to go straight at it, at right angles. That is the direction of the normal, whose numbers are the coefficients in the plane's equation.

Normal: $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$



$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x = 4 + t \qquad y = 5 + 2t \qquad z = 7 + 2t$$

Step 2. Find where this line meets the plane.

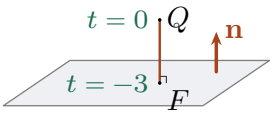
This is Method 2: substitute x, y, z into the plane's equation and solve.

$$(4 + t) + 2(5 + 2t) + 2(7 + 2t) = 1$$
$$4 + t + 10 + 4t + 14 + 4t = 1$$
$$28 + 9t = 1 \qquad t = -3$$

Step 3. Substitute t into x, y and z.

This gives the closest point.

$$x = 4 + (-3) = 1 \qquad y = 5 + 2(-3) = -1$$
$$z = 7 + 2(-3) = 1 \qquad F(1, -1, 1)$$



Check *F should fit the plane's equation.*

$$1 + 2(-1) + 2(1) = 1$$

Step 4. Find the length of QF.

$\overrightarrow{QF}$ is F minus Q. Its length is the distance.

$$\overrightarrow{QF} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -6 \end{pmatrix}$$

$$QF = \sqrt{9 + 36 + 36} = \sqrt{81} \qquad QF = 9$$

If you only need the distance: use the formula booklet. Write the plane with everything on one side, $x + 2y + 2z - 1 = 0$. The distance from (α, β, γ) to $n_1x + n_2y + n_3z + d = 0$ is

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \qquad \text{which here is} \qquad \frac{|4 + 10 + 14 - 1|}{\sqrt{1 + 4 + 4}} = \frac{27}{3} = 9$$

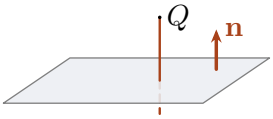
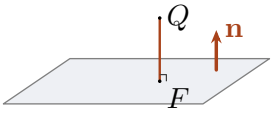
The next page shows where this formula comes from.

Where the formula booklet's distance formula comes from

The formula booklet says: *the perpendicular distance from* (α, β, γ) *to* $n_1x + n_2y + n_3z + d = 0$ *is*

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

It is Method 5 carried out with letters in place of numbers.

The method	With letters
Step 1. Write the line through the point in the direction of the normal. The normal's numbers are the coefficients n_1, n_2, n_3.	$\mathbf{r} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} + t\begin{pmatrix}n_1\\ n_2\\ n_3\end{pmatrix}$ $x = \alpha + tn_1 \quad y = \beta + tn_2 \quad z = \gamma + tn_3$ 
Step 2. Substitute into the plane's equation.	$n_1(\alpha + tn_1) + n_2(\beta + tn_2) + n_3(\gamma + tn_3) + d = 0$
Step 3. Collect the terms with t. Expand the brackets and group.	$(n_1\alpha + n_2\beta + n_3\gamma + d) + t\,(n_1^2 + n_2^2 + n_3^2) = 0$
Step 4. Solve for t. This is the number of steps of $\mathbf{n}$ needed to reach the plane.	$t = -\dfrac{n_1\alpha + n_2\beta + n_3\gamma + d}{n_1^2 + n_2^2 + n_3^2}$ 
Step 5. Multiply the number of steps by the length of one step. One step is the vector $\mathbf{n}$. A distance is positive, so use the size of t, written $\lvert t\rvert$.	distance $= \lvert t\rvert \times \lvert\mathbf{n}\rvert$ where $\lvert t\rvert = \dfrac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d\rvert}{n_1^2 + n_2^2 + n_3^2}$ $\lvert\mathbf{n}\rvert = \sqrt{n_1^2 + n_2^2 + n_3^2}$
Step 6. Simplify. Dividing by $n_1^2 + n_2^2 + n_3^2$ and multiplying by its square root leaves one square root on the bottom.	distance $= \dfrac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d\rvert}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$

Using it: the plane must be written with everything on one side. For example $x+2y+2z = 1$ becomes $x + 2y + 2z - 1 = 0$, so $n_1 = 1$, $n_2 = 2$, $n_3 = 2$ and $d = -1$.

Method 6 Reflect a point in a plane

Use this to find the mirror image of a point in a plane.

Example Find the image Q' of $Q(4,5,7)$ under reflection in the plane $x+2y+2z=1$.

The method

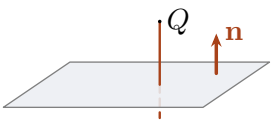
In the example

Step 1. Write the line through Q in the direction of the normal.

The image is straight through the mirror, so it is on this line.

$$\mathbf{r}=\begin{pmatrix}4\\5\\7\end{pmatrix}+t\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$x=4+t \qquad y=5+2t \qquad z=7+2t$$

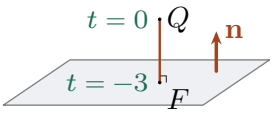


Step 2. Find the value of t where this line meets the plane.

Substitute into the plane's equation and solve (Method 2). This is the point F.

$$(4+t)+2(5+2t)+2(7+2t)=1$$

$$28+9t=1 \qquad t=-3$$



Step 3. Double that value of t.

Q is at $t=0$. The plane is reached after t steps, and the image is the same distance again on the far side.

plane at $t=-3$

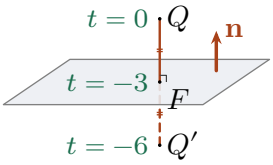
↓ twice as far

image at $t=-6$

Step 4. Substitute the doubled value into x, y and z.

$$x=4+(-6)=-2 \qquad y=5+2(-6)=-7$$

$$z=7+2(-6)=-5 \qquad Q'(-2,-7,-5)$$



Check *The point halfway between Q and Q' should be on the plane.*

Halfway: $\left(\frac{4-2}{2},\ \frac{5-7}{2},\ \frac{7-5}{2}\right)=(1,-1,1)$

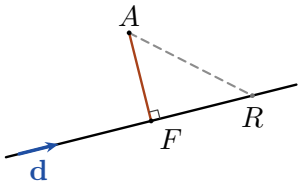
$$1+2(-1)+2(1)=1$$

Method 7 Find the distance from a point to a line

Use this to find the closest point on a line to a given point, and how far away it is.

Example Find the point F on this line that is closest to $A(5,1,6)$, and the distance AF.

$$\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}1\\0\\1\end{pmatrix}$$



The method	In the example
Step 1. Write a general point R on the line. Read across each row of the line's equation.	$R\,(1+\lambda,\ 1,\ \lambda)$
Step 2. Write the vector from A to R. This is R minus A. It still contains λ, because R can be anywhere on the line.	$\overrightarrow{AR} = \begin{pmatrix}1+\lambda\\1\\\lambda\end{pmatrix} - \begin{pmatrix}5\\1\\6\end{pmatrix} = \begin{pmatrix}\lambda-4\\0\\\lambda-6\end{pmatrix}$
Step 3. Make this vector perpendicular to the line. As R slides along the line, AR is shortest when it meets the line at right angles. So its scalar product with the line's direction is zero.	$\begin{pmatrix}\lambda-4\\0\\\lambda-6\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 0$ $(\lambda-4)(1)+(0)(0)+(\lambda-6)(1)=0$ $2\lambda-10=0 \qquad \lambda=5$
Step 4. Substitute λ into R. This gives the closest point.	$F\,(1+5,\ 1,\ 5) \qquad F(6,1,5)$
Step 5. Write the vector from A to F. This is F minus A. It is the perpendicular from A to the line.	$\overrightarrow{AF} = \begin{pmatrix}6\\1\\5\end{pmatrix} - \begin{pmatrix}5\\1\\6\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$
Check *The perpendicular should be at right angles to the line's direction.*	$\begin{pmatrix}1\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 1+0-1=0$
Step 6. Find the length of AF.	$AF = \sqrt{1^2+0^2+(-1)^2} \qquad AF=\sqrt{2}$

To reflect A in the line: go from A to F, then the same again: $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AF}$, here giving $A'(7,1,4)$.

Points on the line at a given distance from A: use Pythagoras in triangle AFB to find FB, then move that far along the line from F each way.

Method 8 Project a line onto a plane

Use this to find the shadow (projection) of a line on a plane.

Example Find the projection of this line onto this plane.

$$l:\ \mathbf{r} = \begin{pmatrix}-1\\1\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad x + 2y + 2z = 1$$

The method

In the example

Step 1. Find P, where the line meets the plane.

This is Method 2. P is already on the plane, so it is on the projection.

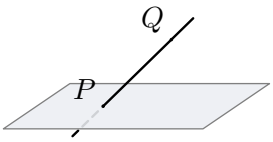
$$x = -1 + \lambda \qquad y = 1 \qquad z = -3 + \lambda$$
$$(-1 + \lambda) + 2(1) + 2(-3 + \lambda) = 1$$
$$-5 + 3\lambda = 1 \qquad \lambda = 2$$
$$P\,(-1 + 2,\ 1,\ -3 + 2) = (1, 1, -1)$$

Step 2. Choose any other point Q on the line.

A line is fixed by two points, so one more shadow point is enough.

Choosing $\lambda = 5$:

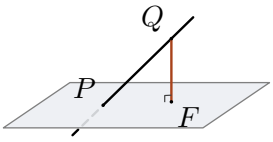
$$Q\,(-1 + 5,\ 1,\ -3 + 5) = (4, 1, 2)$$



Step 3. Find F, the point of the plane closest to Q.

This is Method 5: travel from Q along the normal until you reach the plane. F is the shadow of Q.

$$\mathbf{r} = \begin{pmatrix}4\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$$
$$(4 + t) + 2(1 + 2t) + 2(2 + 2t) = 1$$
$$10 + 9t = 1 \qquad t = -1$$
$$F\,(4 + (-1),\ 1 + 2(-1),\ 2 + 2(-1)) = (3, -1, 0)$$

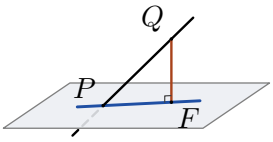


Step 4. Write the line through P and F.

Its direction is $\overrightarrow{PF}$, which is F minus P.

$$\overrightarrow{PF} = \begin{pmatrix}3\\-1\\0\end{pmatrix} - \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$$



Check *The projection lies in the plane, so its direction should be at right angles to the normal.*

$$\begin{pmatrix}2\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = 2 - 4 + 2 = 0$$

Also use it when a question asks for the line in the plane through P that makes the same angle with l as the plane does, or the smallest possible angle with l: that line is the projection.

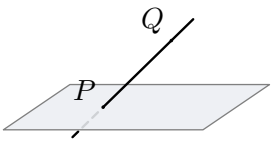
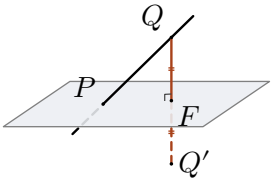
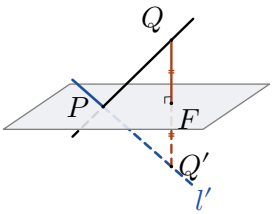
If the line is parallel to the plane: there is no P. Use one shadow point with the line's own direction.

Method 9 Reflect a line in a plane

Use this to find the mirror image of a line in a plane.

Example Find the image l' of this line under reflection in this plane.

$$l:\ \mathbf{r} = \begin{pmatrix}-1\\1\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad x + 2y + 2z = 1$$

The method	In the example
Step 1. Find P, where the line meets the plane. This is Method 2. P is in the mirror, so it does not move: it is also on l'.	$(-1+\lambda) + 2(1) + 2(-3+\lambda) = 1$ $-5 + 3\lambda = 1 \qquad \lambda = 2$ $P\,(-1+2,\ 1,\ -3+2) = (1, 1, -1)$
Step 2. Choose any other point Q on the line. Pick any other value of λ.	Choosing $\lambda = 5$: $\quad Q(4, 1, 2)$ 
Step 3. Reflect Q in the plane to get Q'. This is Method 6: travel from Q along the normal, find t at the plane, then double it.	$\mathbf{r} = \begin{pmatrix}4\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$ $(4+t) + 2(1+2t) + 2(2+2t) = 1$ $10 + 9t = 1 \qquad$ plane at $t = -1$ $\downarrow$ twice as far: $t = -2$ $Q'\,(4+(-2),\ 1+2(-2),\ 2+2(-2)) = (2, -3, -2)$ 
Step 4. Write the line through P and Q'. Its direction is $\overrightarrow{PQ'}$, which is Q' minus P.	$\overrightarrow{PQ'} = \begin{pmatrix}2\\-3\\-2\end{pmatrix} - \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\-4\\-1\end{pmatrix}$ $\mathbf{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\-4\\-1\end{pmatrix}$ 
Check *The point halfway between Q and Q' should be on the plane.*	Halfway: $(3, -1, 0) \qquad 3 + 2(-1) + 2(0) = 1$

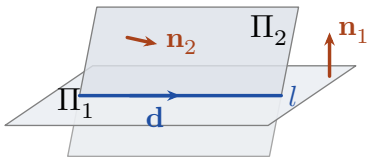
If the line is parallel to the plane: there is no point P. Reflect one point of the line, and use it with the line's original direction.

Method 10 Find the line of intersection of two planes

Use this to find the line where two planes cross.

Example Find the line of intersection of these planes.

$$x + y + 2z = 5 \qquad 2x + y + z = 4$$



The method

In the example

Step 1. Let one coordinate be the parameter.

Points on the line fit both equations. Two equations cannot fix three unknowns, so one is left free to vary. That free one becomes the parameter of the line.

Let $z = t$.

$$x + y + 2t = 5 \quad \text{so} \quad x + y = 5 - 2t$$
$$2x + y + t = 4 \quad \text{so} \quad 2x + y = 4 - t$$

Step 2. Solve for the other two coordinates.

Treat t as if it were a known number and solve the simultaneous equations.

Subtracting the first from the second:

$$x = (4 - t) - (5 - 2t) = -1 + t$$

Then from the first:

$$y = 5 - 2t - x = 5 - 2t - (-1 + t) = 6 - 3t$$

Step 3. Write x, y, z as a column and split off the t part.

The numbers without t give a point on the line. The numbers multiplying t give its direction.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 + t \\ 6 - 3t \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

Check *The line lies in both planes, so its direction should be at right angles to both normals.*

$$\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 - 3 + 2 = 0 \qquad \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 - 3 + 1 = 0$$

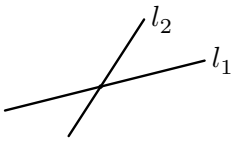
If Step 2 will not solve: start again with a different coordinate as the parameter. (If one normal is a multiple of the other, the planes are parallel and there is no line to find.)

Method 11 Decide whether two lines meet

Use this to find the point where two lines cross, or show that they do not meet.

Example Do these lines meet? If so, where?

$$\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix}0\\1\\4\end{pmatrix} + \mu \begin{pmatrix}1\\0\\-1\end{pmatrix}$$



The method	In the example
Step 1. Write a general point on each line. Use a different letter for each line's parameter. The meeting point can be a different number of steps along each line.	first line: $(1+\lambda,\ \lambda,\ 2)$ second line: $(\mu,\ 1,\ 4-\mu)$
Step 2. Set the two points equal. If the lines meet, these are the same point, so the x, y and z coordinates all match.	$x:\quad 1+\lambda=\mu$ $y:\quad \lambda=1$ $z:\quad 2=4-\mu$
Step 3. Solve two of the equations. Two equations are enough to find λ and μ.	From y: $\lambda=1$ Into x: $\mu=1+1$, so $\mu=2$
Step 4. Test the values in the third equation. If it works, the lines meet. If not, they do not.	$z:\quad 4-2=2$ This is correct, so the lines meet.
Step 5. Substitute λ into line 1. This gives the point where the lines meet.	$(1+1,\ 1,\ 2)$ $\quad (2,1,2)$

To show that two lines are skew: show that the third equation fails, so they do not meet, and also that the directions are not multiples of each other, so they are not parallel.

Method 12 Find the distance between two skew lines

Use this to find the shortest distance between two lines that neither meet nor are parallel.

Example Find the shortest distance between these lines.

$$l_1:\ \mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix} \qquad l_2:\ \mathbf{r}=\begin{pmatrix}2\\6\\2\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\0\end{pmatrix}$$

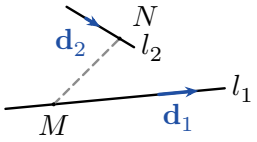
The method

In the example

Step 1. Write a general point on each line.

M is on l_1 and N is on l_2.

$$M\,(1+\lambda,\ 1,\ 1-\lambda)$$
$$N\,(2+\mu,\ 6-2\mu,\ 2)$$



Step 2. Write the vector from M to N.

This is N minus M.

$$\overrightarrow{MN}=\begin{pmatrix}1+\mu-\lambda\\5-2\mu\\1+\lambda\end{pmatrix}$$

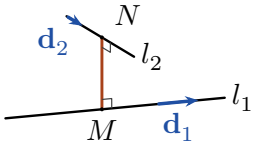
Step 3. Make it perpendicular to both lines.

The shortest link between the lines meets both at right angles. So its scalar product with each direction is zero.

$$\overrightarrow{MN}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=0 \text{ gives } (1+\mu-\lambda)-(1+\lambda)=0$$

$$\overrightarrow{MN}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=0 \text{ gives } (1+\mu-\lambda)-2(5-2\mu)=0$$

which simplify to $\mu-2\lambda=0$ and $5\mu-\lambda=9$



Step 4. Solve the two equations.

From the first, $\mu=2\lambda$. Into the second:

$$10\lambda-\lambda=9 \qquad \lambda=1 \qquad \mu=2$$

Step 5. Substitute to find M, N and $\overrightarrow{MN}$.

$\overrightarrow{MN}$ is now the perpendicular to both lines.

$$M\,(1+1,\ 1,\ 1-1)=(2,1,0)$$
$$N\,(2+2,\ 6-2(2),\ 2)=(4,2,2)$$
$$\overrightarrow{MN}=\begin{pmatrix}2\\1\\2\end{pmatrix}$$

Check *The perpendicular should be at right angles to both directions.*

$$\overrightarrow{MN}\cdot\mathbf{d}_1=2+0-2=0 \qquad \overrightarrow{MN}\cdot\mathbf{d}_2=2-2+0=0$$

Step 6. Find the length of MN.

$$MN=\sqrt{4+1+4} \qquad MN=3$$

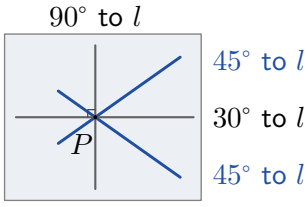
Also: the line meeting both at right angles is the line through M and N. The plane containing l_1 and parallel to l_2 has $\overrightarrow{MN}$ as its normal.

Method 13 Find a line in a plane at a given angle

Use this to find a line that lies in a given plane, passes through a given point, and makes a given angle with another line.

Example A line l meets the plane $x + z = 3$ at $P(1, 2, 2)$. Find the lines in the plane through P that make an angle of $45°$ with l.

direction of l : $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$



view from directly above
angles are with l, not as seen here

The method

In the example

Step 1. Call the unknown direction $\mathbf{b}$, and use "lies in the plane".

P is known, so only the direction is missing. A line in the plane is at right angles to the plane's normal, so the scalar product is zero.

Let $\mathbf{b}$ have components p, q, r. The plane has normal $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$ gives $p + r = 0$, so $r = -p$

Step 2. Use the angle and square it.

The angle formula has a modulus and square roots. Squaring both sides removes them. Use the exact value $\cos^2 45° = \frac{1}{2}$.

$$(\mathbf{b} \cdot \mathbf{d})^2 = |\mathbf{b}|^2 \, |\mathbf{d}|^2 \cos^2 45°$$

Here $\mathbf{b} \cdot \mathbf{d} = p - q$, $|\mathbf{b}|^2 = p^2 + q^2 + r^2$ and $|\mathbf{d}|^2 = 2$:

$$(p - q)^2 = (p^2 + q^2 + r^2)(2)\left(\tfrac{1}{2}\right) = p^2 + q^2 + r^2$$

Step 3. Substitute and solve.

Use the result of Step 1 to remove one letter, then solve what is left.

Putting $r = -p$, so that $r^2 = p^2$:

$$p^2 - 2pq + q^2 = p^2 + q^2 + p^2$$

$-2pq = p^2$ $\quad p(p + 2q) = 0$ $\quad p = 0$ or $p = -2q$

Step 4. Turn each answer into a direction.

A direction can be any length, so choose a simple value for one letter.

If $p = 0$: $r = 0$, and taking $q = 1$ gives $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

If $p = -2q$: taking $q = -1$ gives $p = 2$, $r = -2$, so $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Check *The angle between each answer and l should be $45°$.*

$$\frac{|0 - 1 + 0|}{1 \times \sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \frac{|2 + 1 + 0|}{3 \times \sqrt{2}} = \frac{1}{\sqrt{2}}$$

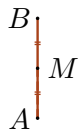
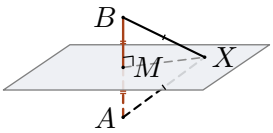
both give $45°$

How many answers: if θ is the angle between l and the plane (and l is not perpendicular to the plane), an angle below θ gives none; equal to θ gives one (the projection); between θ and $90°$ gives two; $90°$ gives one.

Method 14 Find the plane equidistant from two points

Use this to find the plane made of all the points that are the same distance from two given points.

Example A is $(1, 0, 2)$ and B is $(3, 4, 0)$. Find the plane of points X for which $AX = BX$.

The method	In the example
Step 1. Find the midpoint of AB. The midpoint is the same distance from A and B, so it is on the plane. Average the coordinates.	$M\left(\frac{1+3}{2}, \frac{0+4}{2}, \frac{2+0}{2}\right)$ $\quad M(2, 2, 1)$ 
Step 2. Use $\overrightarrow{AB}$ as the normal. The plane cuts AB at right angles, so $\overrightarrow{AB}$ sticks straight out of it. Any multiple will do, so halve it to make the numbers smaller.	$\overrightarrow{AB} = \begin{pmatrix}3\\4\\0\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\4\\-2\end{pmatrix}$ $\mathbf{n} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$ 
Step 3. Write the plane using the normal. The normal's numbers are the coefficients of x, y, z.	$1x + 2y - 1z = k$
Step 4. Substitute the midpoint to find k.	Using $M(2, 2, 1)$: $k = 2 + 2(2) - 1 = 5$ $x + 2y - z = 5$
Check *Any point on the plane should be equally far from A and B.*	$X(5, 0, 0)$ is on the plane. $AX^2 = 4^2 + 0^2 + 2^2 = 20$ $BX^2 = 2^2 + 4^2 + 0^2 = 20$

With algebra instead: let X be (x, y, z) and write $AX^2 = BX^2$ using the distance formula. When you expand the brackets, the x^2, y^2 and z^2 terms cancel, leaving the equation of a plane. Use this version if the question tells you to start from (x, y, z).

For a condition like $AX^2 - BX^2 = k$: use the algebra. The squared terms still cancel, and the result is a plane parallel to the one above.

Method 15 Find the shortest path via a plane

Use this to find the point X on a plane that makes the total distance $AX + XB$ as small as possible, when A and B are on the same side of the plane.

Example A is $(4, 5, 7)$ and B is $(4, 7, 5)$. Find the point X on the plane $x + 2y + 2z = 1$ for which $AX + XB$ is least, and find the least value.

The method

In the example

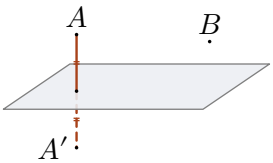
Step 1. Reflect A in the plane to get A'.

This is Method 6. Every point of the plane is the same distance from A as from A', so $AX + XB$ equals $A'X + XB$.

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad 28 + 9t = 1$$

Plane at $t = -3$, so image at $t = -6$:

$$A'(-2, -7, -5)$$



Step 2. Write the line through A' and B.

A journey from A' to B is shortest when it is a straight line. Halve the direction to make the numbers smaller.

$$\overrightarrow{A'B} = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 10 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} -2 \\ -7 \\ -5 \end{pmatrix} + s \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$$

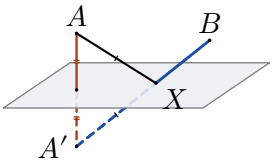
Step 3. Find where this line meets the plane.

This is Method 2. The meeting point is X.

$$(-2 + 3s) + 2(-7 + 7s) + 2(-5 + 5s) = 1$$

$$-26 + 27s = 1 \qquad s = 1$$

$$X\,(-2+3(1),\ -7+7(1),\ -5+5(1)) \qquad X(1, 0, 0)$$



Check *X should fit the plane's equation.*

$$1 + 2(0) + 2(0) = 1$$

Step 4. Find the length of $A'B$.

This straight-line distance is the least total distance.

$$A'B = \sqrt{6^2 + 14^2 + 10^2} = \sqrt{332}$$

$$AX + XB = 2\sqrt{83}$$

Method 16 Find a line through a point that meets two lines

Use this to find the line through a given point A that crosses two given lines.

Example Find the line through $A(1,1,1)$ that meets both of these lines.

$$l_1:\ \mathbf{r}=\begin{pmatrix}5\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\2\end{pmatrix} \qquad l_2:\ \mathbf{r}=\begin{pmatrix}-4\\3\\0\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

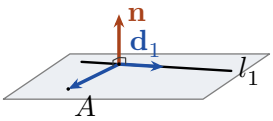
The method

In the example

Step 1. Find the plane containing A and l_1.

This is Method 1. Any line from A to a point of l_1 stays in this plane, so the line we want lies in it.

Directions in the plane: the direction of l_1, and the vector to A from the point $(5,1,-1)$ on l_1.



$$\begin{pmatrix}0\\1\\2\end{pmatrix} \qquad \begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}5\\1\\-1\end{pmatrix}=\begin{pmatrix}-4\\0\\2\end{pmatrix}$$

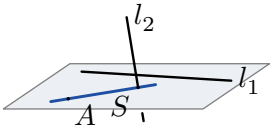
$$b+2c=0 \qquad -4a+2c=0$$

Taking $a=1$: $c=2$, $b=-4$, so $\mathbf{n}=\begin{pmatrix}1\\-4\\2\end{pmatrix}$

$$x-4y+2z=k \qquad k=1-4+2=-1$$

Step 2. Find S, where l_2 meets this plane.

This is Method 2. The line we want meets l_2 and lies in the plane, so it has to pass through S.



$$x=-4+\mu \qquad y=3-\mu \qquad z=0$$

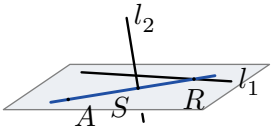
$$(-4+\mu)-4(3-\mu)+2(0)=-1$$

$$-16+5\mu=-1 \qquad \mu=3$$

$$S\,(-4+3,\ 3-3,\ 0)=(-1,0,0)$$

Step 3. Write the line through A and S.

Its direction is $\overrightarrow{SA}$, which is A minus S.



$$\overrightarrow{SA}=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Step 4. Check that this line meets l_1.

This is Method 11. Both lines are in the same plane, so they meet unless they are parallel. The check also gives the meeting point.

$$(1+2s,\ 1+s,\ 1+s)=(5,\ 1+\lambda,\ -1+2\lambda)$$

$$x:\ 1+2s=5\ \Rightarrow\ s=2 \qquad y:\ 1+s=1+\lambda\ \Rightarrow\ \lambda=2$$

$$z:\ 1+s=-1+2\lambda \text{ gives } 3=3$$

All three agree, so the lines meet at $R(5,3,3)$.

When there is no such line: if l_2 is parallel to the plane in Step 2 (and not in it), there is no point S. If the line in Step 3 is parallel to l_1, it never meets l_1. In both cases no line through A meets both.

Which methods does each question use?

Try each question first. Use this table if you are stuck, or afterwards to check.

Methods 1 to 5, 7 and 10 to 12 are the standard building blocks. The others apply them to particular situations, so follow how each one is built rather than memorising it as a separate procedure.

Question	Methods	Method	
1	2, 4, 5	**1**	Find the Cartesian equation of a plane
2	1, 3	**2**	Find where a line meets a plane
3	2, 6, 15	**3**	Find the angle between two planes (or two lines)
4	7	**4**	Find the angle between a line and a plane
5	2, 4, 8	**5**	Find the distance from a point to a plane
6	2, 9	**6**	Reflect a point in a plane
7	1, 2, 3, 10	**7**	Find the distance from a point to a line
8	2, 5, 9	**8**	Project a line onto a plane
9	11, 12	**9**	Reflect a line in a plane
10	2, 4, 8, 13	**10**	Find the line of intersection of two planes
11	2, 14	**11**	Decide whether two lines meet
12	1, 2, 11, 16	**12**	Find the distance between two skew lines
		13	Find a line in a plane at a given angle
		14	Find the plane equidistant from two points
		15	Find the shortest path via a plane
		16	Find a line through a point that meets two lines