Vectors Practice - Worked Solutions

Question 1

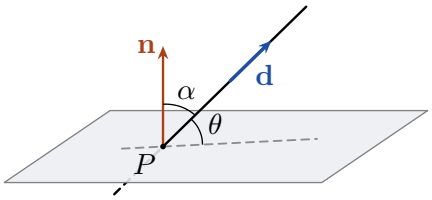
(a) Write l with a parameter: $x = 3 + t$, $y = 1 + 2t$, $z = -1 + 2t$. A point of l is on Π when these satisfy the plane's equation:

$$2(3 + t) - (1 + 2t) + 2(-1 + 2t) = 15 \;\Rightarrow\; 3 + 4t = 15 \;\Rightarrow\; t = 3$$

So $P(6, 7, 5)$.

(b) The line's direction $\mathbf{d}$ and the plane's normal $\mathbf{n}$ are

$$\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$



Their scalar product gives the angle α between the line and the *normal*:

$$\cos\alpha = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \frac{|2 - 2 + 4|}{3 \times 3} = \frac{4}{9} \;\Rightarrow\; \alpha = 63.6^\circ$$

The normal is at right angles to the plane, so the angle θ between the line and the plane (the dashed line) is what is left of the right angle:

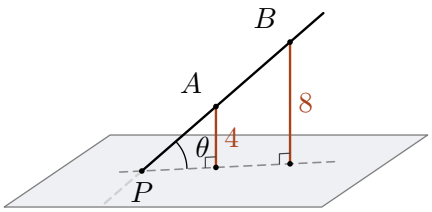
$$\theta = 90^\circ - 63.6^\circ = 26.4^\circ$$

(c) Using the distance formula with $A(3, 1, -1)$:

$$\frac{|2(3) - (1) + 2(-1) - 15|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{12}{3} = 4$$

(d) A is at $t = 0$, B is at $t = -3$ and P is at $t = 3$.

So B is 6 steps of $\mathbf{d}$ from P and A is 3 steps: B is twice as far from P as A is, on the same side.



The perpendiculars from A and B to the plane make right-angled triangles with the same angle θ at P, so the triangles are similar. Doubling the distance along the line doubles the height above the plane.

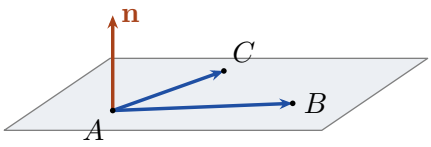
Distance from B to $\Pi = 2 \times 4 = 8$

Question 2

(a) Two directions in the plane are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Starting from A:

$$\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

(b) First find the normal $\mathbf{n}$: the direction at right angles to both blue arrows, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



Call its components a, b and c. A scalar product of zero means perpendicular, so

$$\begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} = 0: \quad a + 2b + c = 0$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}-1\\-1\\1\end{pmatrix} = 0: \quad -a - b + c = 0$$

Adding gives $b + 2c = 0$, so $b = -2c$.

Then $a = -b + c = 3c$.

Choosing $c = 1$:

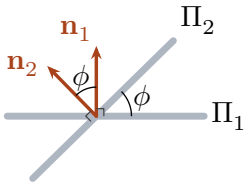
$$\mathbf{n} = \begin{pmatrix}3\\-2\\1\end{pmatrix}$$

So the plane is $3x - 2y + z = k$. The point A lies on it, so

$$k = 3(1) - 2(0) + 1 = 4$$

$$\boxed{3x - 2y + z = 4}$$

(c) The diagram shows two planes seen edge-on. Each normal is at right angles to its plane, so the angle between two planes equals the angle between their normals.



$$\cos\phi = \frac{\left|\begin{pmatrix}3\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\3\end{pmatrix}\right|}{\sqrt{14}\sqrt{14}} = \frac{3+4+3}{14} = \frac{5}{7}$$

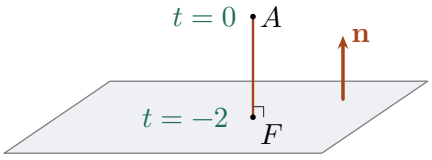
$$\phi = \boxed{44.4^\circ}$$

(d) D must satisfy the plane's equation:

$$3k - 2(2k) + 5 = 4 \;\Rightarrow\; \boxed{k = 1}$$

Question 3

(a) First find the closest point F of the plane to A. It is reached by travelling from A straight towards the plane, along the normal $\mathbf{n}$.



The components of $\mathbf{n}$ are the coefficients 1, 2, -2 in the plane's equation. The line through A along $\mathbf{n}$ is

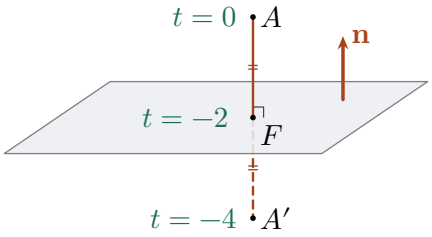
$$\mathbf{r} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

F is where this line meets Π:

$$(3 + t) + 2(6 + 2t) - 2(-3 - 2t) = 3$$

$$21 + 9t = 3 \;\Rightarrow\; t = -2$$

The reflection A' is the same distance beyond the plane, on the same perpendicular.



A is at $t = 0$ and F is at $t = -2$, so A' is at $t = -4$:

$$\begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} + (-4) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

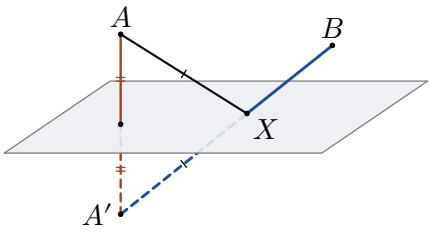
$$A'(-1, -2, 5)$$

(b) Work out $x + 2y - 2z$ at each point. On the plane this expression equals 3; on one side it is greater than 3 and on the other it is less.

$$\text{At } A: \; 3 + 12 + 6 = 21 \qquad \text{At } B: \; 4 + 6 + 2 = 12$$

Both values are greater than 3, so A and B are on the same side.

(c) Because A' is the mirror image of A, every point X on the plane is the same distance from A as from A'. So



$$AX + XB = A'X + XB$$

The right-hand side is a route from A' to B, which is shortest when it is a straight line. So X is where the line $A'B$ crosses Π.

$$\overrightarrow{A'B} = \begin{pmatrix} 5 \\ 5 \\ -6 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 5 \\ 5 \\ -6 \end{pmatrix}$$

$$(-1 + 5s) + 2(-2 + 5s) - 2(5 - 6s) = 3$$

$$-15 + 27s = 3 \;\Rightarrow\; s = \tfrac{2}{3}$$

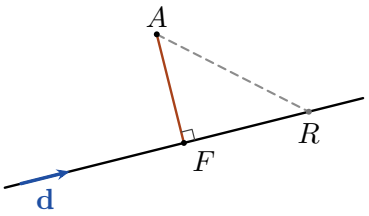
$$X\left(\tfrac{7}{3}, \tfrac{4}{3}, 1\right)$$

The minimum is the length $A'B$:

$$\sqrt{25 + 25 + 36} = \sqrt{86}$$

Question 4

(a) Let R be any point on l:



$$R(1+2\lambda,\ \lambda,\ 2-2\lambda)$$

$$\overrightarrow{AR} = \begin{pmatrix} 2\lambda-6 \\ \lambda-6 \\ -2\lambda \end{pmatrix}$$

As R slides along the line, AR is shortest when it meets the line at right angles. That position of R is F, so:

$$\overrightarrow{AR} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$2(2\lambda-6) + (\lambda-6) - 2(-2\lambda) = 0$$

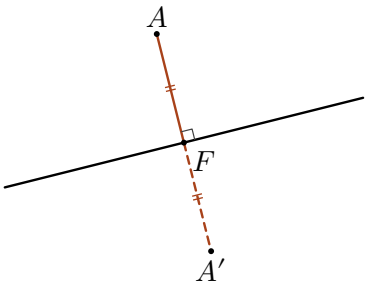
$$9\lambda - 18 = 0 \ \Rightarrow\ \lambda = 2$$

So F is $(5, 2, -2)$ and

$$\overrightarrow{AF} = \begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix}$$

$$AF = \sqrt{4+16+16} = \boxed{6}$$

(b) The reflection A' is found by going from A to F and then the same again, so F is the midpoint of AA'.

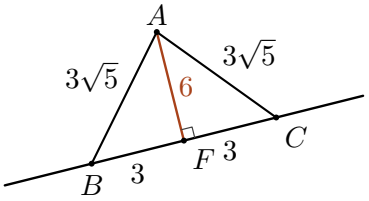


$$\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AF} = \begin{pmatrix} 7 \\ 6 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix}$$

$$\boxed{A'(3, -2, -6)}$$

(c) Triangle AFB is right-angled at F.

$AF = 6$ and $AB = 3\sqrt{5}$, so



$$FB^2 = 45 - 36 = 9 \ \Rightarrow\ FB = 3$$

B and C are 3 units either side of F along the line.

One step of $\mathbf{d}$ has length $|\mathbf{d}| = 3$, so they are one step either side of $\lambda = 2$:

$$\lambda = 1: \ \boxed{(3, 1, 0)} \qquad \lambda = 3: \ \boxed{(7, 3, -4)}$$

(d) The base is $BC = 6$ and the height is $AF = 6$.

$$\text{Area} = \tfrac{1}{2} \times 6 \times 6 = \boxed{18}$$

Question 5

(a) In parametric form, $x = 2 + t$, $y = 2t$, $z = -2 + 3t$. In the plane:

$$(2 + t) + 2t + (-2 + 3t) = 6 \;\Rightarrow\; t = 1$$

So $P(3, 2, 1)$.

(b) Let θ be the angle between l_1 and Π. First find the angle α between l_1 and the normal to Π:

$$\cos\alpha = \frac{\left|\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{14}\sqrt{3}} = \frac{6}{\sqrt{42}} \;\Rightarrow\; \alpha = 22.2^\circ$$

The angle between the line and the plane is what is left of the right angle:

$$\theta = 90^\circ - 22.2^\circ = 67.8^\circ$$

(c) Take any other point Q on l_1, and let F be the point of Π closest to Q.

Now take any line in Π through P, and let G be the point of that line closest to Q. Triangle PGQ is right-angled at G, so the angle ϕ between l_1 and that line has

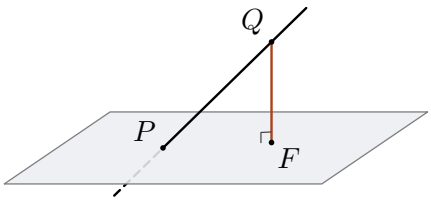
$$\sin\phi = \frac{QG}{PQ}$$

F is the closest point of the whole plane to Q, so QG is never less than QF, and it equals QF only when the line passes through F.

So the smallest angle is made by the line through P and F. This line is the projection (the shadow) of l_1 on Π, and the angle is θ from part **(b)**.

For Q, take $t = 2$, which gives $Q(4, 4, 4)$.

Drop a perpendicular from Q to the plane by travelling along the normal, which has components 1, 1, 1:



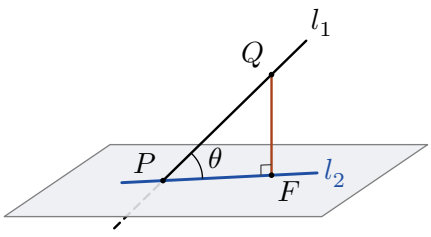
$$\mathbf{r} = \begin{pmatrix}4\\4\\4\end{pmatrix} + s\begin{pmatrix}1\\1\\1\end{pmatrix}$$

This meets Π when

$$12 + 3s = 6 \;\Rightarrow\; s = -2$$

The foot of the perpendicular is $F(2, 2, 2)$.

F is the point of the plane directly "below" Q, so l_2 is the line through P and F.



$$\overrightarrow{PF} = \begin{pmatrix}2\\2\\2\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix}3\\2\\1\end{pmatrix} + \mu\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Question 6

(a) Substitute a general point of l into the plane's equation:

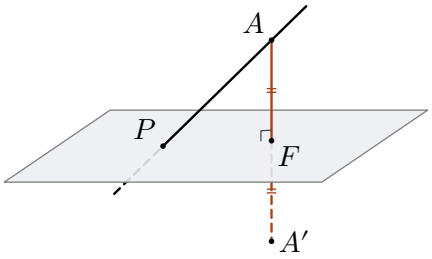
$$2(1+4\lambda)-(2+\lambda)+2\lambda=9 \;\Rightarrow\; 9\lambda=9 \;\Rightarrow\; \lambda=1$$

So $P(5,3,1)$.

(b) Every point of l reflects to a point of l', and two points fix a line. P is in the mirror, so it does not move: l' passes through P. We need to reflect one other point of l.

Take $A(1,2,0)$, the point of l with $\lambda=0$. Travel from A along the normal, which has components 2, -1, 2:

$$\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\-1\\2\end{pmatrix}$$



This reaches the plane when

$$2(1+2s)-(2-s)+2(2s)=9 \;\Rightarrow\; s=1$$

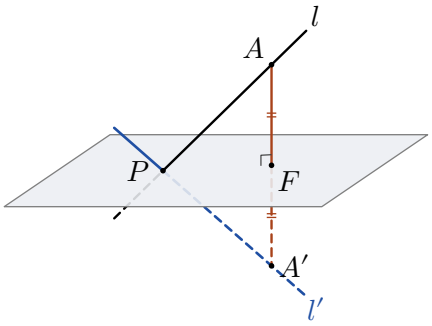
So the foot of the perpendicular F is at $s=1$. The reflection is the same distance again, at $s=2$:

$$A'(5,0,4)$$

l' passes through P and A'.

$$\overrightarrow{PA'}=\begin{pmatrix}5\\0\\4\end{pmatrix}-\begin{pmatrix}5\\3\\1\end{pmatrix}=\begin{pmatrix}0\\-3\\3\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}5\\3\\1\end{pmatrix}+\mu\begin{pmatrix}0\\-1\\1\end{pmatrix}$$



(c) The directions of l and l' have scalar product

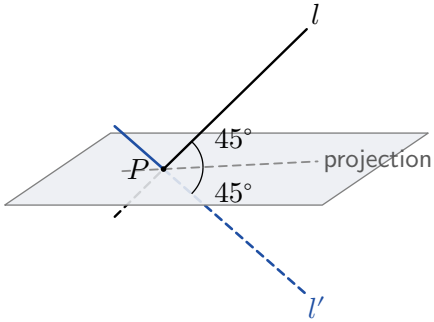
$$\begin{pmatrix}4\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}=0-1+1=0$$

so l and l' are perpendicular.

(d) l and l' are mirror images, so they make the same angle with the plane, one above it and one below.

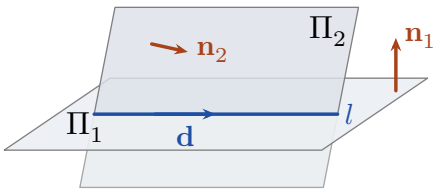
l, l' and the normal at P all lie in one plane at right angles to Π. The dashed line is where that plane cuts Π: it is the projection of both l and l', and it lies exactly between them.

The two equal angles add up to the $90°$ between l and l', so each is $45°$.



Question 7

(a) Points on l satisfy both plane equations at once. Two equations in three unknowns leave one unknown free, so let $z = t$ and solve for the other two:



$$x + 2y = 4 + t$$

$$2x - y = 3 - 3t$$

Adding the first to twice the second:

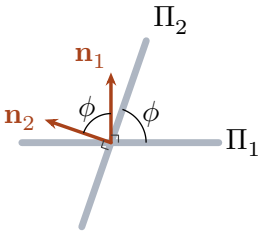
$$5x = 10 - 5t \;\Rightarrow\; x = 2 - t$$

$$y = 2x - 3 + 3t = 1 + t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - t \\ 1 + t \\ t \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(b) The diagram shows the two planes edge-on, looking along l. The angle between the planes equals the angle between their normals:

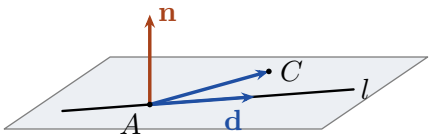


$$\cos\phi = \frac{\left|\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\right|}{\sqrt{6}\sqrt{14}} = \frac{3}{\sqrt{84}}$$

$$\phi = 70.9^\circ$$

(c) To find a plane we need two directions in it.

One is the direction $\mathbf{d}$ of l. For the other, take the point $A(2, 1, 0)$ on l and join it to $C(1, 3, 2)$.



$$\mathbf{d} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Let the normal have components a, b, c. It is perpendicular to both:

$$-a + b + c = 0$$

$$-a + 2b + 2c = 0$$

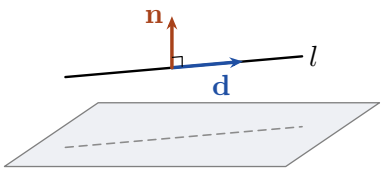
Subtracting gives $b + c = 0$, so $c = -b$, and then $a = 0$.

Taking $b = 1$, the normal has components 0, 1, -1 and the plane is $y - z = k$.

C is on the plane, so $k = 3 - 2 = 1$.

$$y - z = 1$$

(d) Π_4 has normal $\mathbf{n}$ and l has direction $\mathbf{d}$, where



$$\mathbf{n} = \begin{pmatrix} 1 \\ 7 \\ -6 \end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{n} \cdot \mathbf{d} = -1 + 7 - 6 = 0$$

The direction of l is at right angles to the normal, which means l never crosses the plane: it either runs parallel to it or lies in it. This does not involve k, so it is true for every k.

(e) A line that does not cross a plane lies in it if any one of its points does. Using the point $(2, 1, 0)$ of l:

$$k = 2 + 7(1) - 6(0) \;\Rightarrow\; \boxed{k = 9}$$

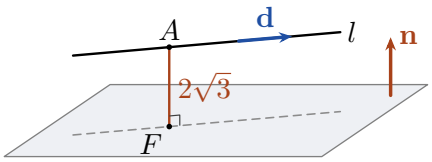
Question 8

(a) Substitute a general point of l into the plane's equation:

$$(1+2\lambda)-(4+\lambda)+(2-\lambda)=-1$$

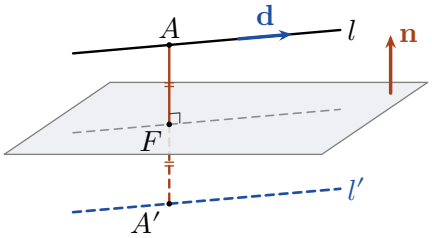
The λ terms cancel, so every point of l gives -1, never 5. No point of l is on Π.

(b) The λ terms cancelled because l is parallel to Π. So every point of l is the same distance from the plane, and we can use any one point, such as $A(1,4,2)$:



$$\frac{|1-4+2-5|}{\sqrt{1^2+1^2+1^2}}=\frac{6}{\sqrt{3}}=2\sqrt{3}$$

(c) The mirror image of a line parallel to the mirror is another parallel line, so l' has the same direction $\mathbf{d}$. We only need one point of it: the reflection of A.



Travel from A along the normal:

$$\mathbf{r}=\begin{pmatrix}1\\4\\2\end{pmatrix}+s\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

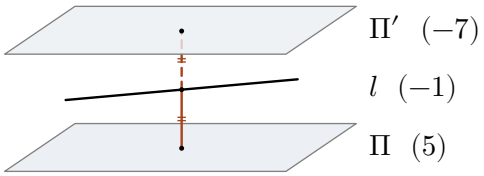
This meets Π when

$$-1+3s=5 \;\Rightarrow\; s=2$$

So F is at $s=2$ and A' is at $s=4$, which is $(5,0,6)$.

$$\mathbf{r}=\begin{pmatrix}5\\0\\6\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

(d) Planes parallel to Π have the same normal, so they are $x-y+z=c$ for different values of c. Changing c slides the plane along its normal at a steady rate.



The diagram shows the value of $x-y+z$ on each object in brackets. Every point of l has value -1, and Π has $c=5$, which is 6 greater.

Equal changes in c correspond to equal perpendicular distances, so the plane equally far on the other side has c equal to 6 less than -1:

$$x-y+z=-7$$

Question 9

(a) The directions of the lines are not multiples of each other, so the lines are not parallel:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

If the lines met, one point would be on both, so for some λ and μ:

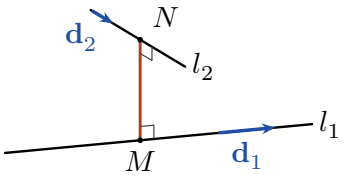
$$1 + 2\lambda = 1 + 2\mu \qquad 1 - \lambda = 2 + \mu \qquad 1 = 9 - 2\mu$$

The first gives $\lambda = \mu$, and then the second gives $\lambda = \mu = -\frac{1}{2}$.

But then the third would say $1 = 10$. No values work, so the lines do not meet.

Not parallel and not meeting means skew.

(b) The shortest link between the lines meets both at right angles. Let it join M on l_1 to N on l_2:



$$M(1 + 2\lambda,\ 1 - \lambda,\ 1)$$

$$N(1 + 2\mu,\ 2 + \mu,\ 9 - 2\mu)$$

$$\overrightarrow{MN} = \begin{pmatrix} 2\mu - 2\lambda \\ 1 + \mu + \lambda \\ 8 - 2\mu \end{pmatrix}$$

MN is perpendicular to l_1:

$$\overrightarrow{MN} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \ \Rightarrow\ 3\mu - 5\lambda = 1$$

MN is perpendicular to l_2:

$$\overrightarrow{MN} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0 \ \Rightarrow\ 3\mu - \lambda = 5$$

Subtracting gives $4\lambda = 4$, so $\lambda = 1$ and $\mu = 2$.

So M is $(3, 0, 1)$ and N is $(5, 4, 5)$.

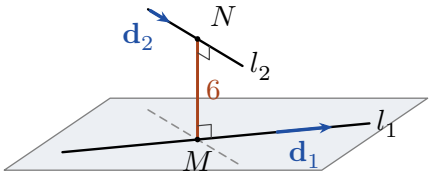
$$\overrightarrow{MN} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$MN = \sqrt{4 + 16 + 16} = 6$$

(c) This is the line through M and N. Halving $\overrightarrow{MN}$ gives a simpler direction:

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(d) The plane contains the direction $\mathbf{d}_1$ and runs parallel to $\mathbf{d}_2$ (the dashed line is l_2 slid down into the plane).



So its normal is perpendicular to both $\mathbf{d}_1$ and $\mathbf{d}_2$. We already have such a vector: $\overrightarrow{MN}$. Halving it, take the normal with components 1, 2, 2.

Using the point $(1, 1, 1)$ on l_1:

$$1 + 2 + 2 = 5$$

$$x + 2y + 2z = 5$$

(e) l_2 is parallel to Π, so all of its points are the same distance from Π. At N that distance is MN, because M is in the plane and MN is perpendicular to it. So the distance is $MN = 6$.

Question 10

(a) Substitute a general point of l_1 into the plane's equation:

$$2(1) + \lambda + 2(1 + \lambda) = 10 \;\Rightarrow\; \lambda = 2$$

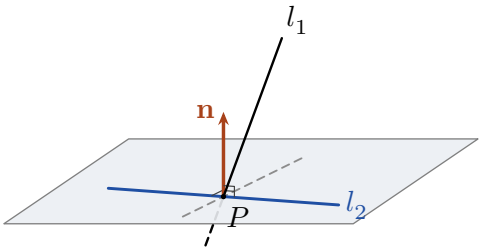
So $P(1, 2, 3)$.

(b) First find the angle α between l_1 and the normal to Π:

$$\cos\alpha = \frac{\left|\begin{pmatrix}0\\1\\1\end{pmatrix} \cdot \begin{pmatrix}2\\1\\2\end{pmatrix}\right|}{\sqrt{2} \times 3} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \;\Rightarrow\; \alpha = 45^\circ$$

The angle between the line and the plane is what is left of the right angle: $\theta = 90^\circ - 45^\circ = 45^\circ$.

(c) Let the direction of l_2 have components p, q, r. Two facts pin it down.



l_2 lies in the plane, so it is at right angles to the normal:

$$2p + q + 2r = 0$$

l_2 is perpendicular to l_1:

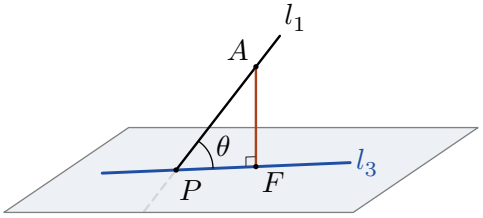
$$q + r = 0$$

From the second, $r = -q$. Then the first gives $2p = q$.

Taking $p = 1$ gives $q = 2$ and $r = -2$.

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

(d) 45° is the angle between l_1 and the plane, which is the smallest angle l_1 makes with any line in the plane. Only one line achieves it: the projection of l_1 (the solution to Question 5(c) shows why). So l_3 is the projection.



Take $A(1, 0, 1)$ on l_1 and drop a perpendicular:

$$\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + s\begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$4 + 9s = 10 \;\Rightarrow\; s = \tfrac{2}{3}$$

So the foot is $F\left(\frac{7}{3}, \frac{2}{3}, \frac{7}{3}\right)$.

$$\overrightarrow{PF} = \begin{pmatrix}4/3\\-4/3\\-2/3\end{pmatrix} = -\tfrac{2}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}-2\\2\\1\end{pmatrix}$$

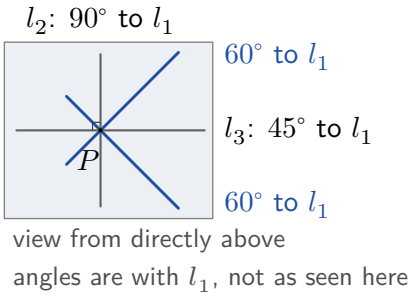
(e) Let the direction be $\mathbf{b}$, with components p, q, r.

In the plane:

$$2p + q + 2r = 0 \;\Rightarrow\; q = -2p - 2r$$

At 60° to l_1, whose direction $\mathbf{d}$ has components 0, 1, 1:

$$\frac{|\mathbf{b}\cdot\mathbf{d}|}{|\mathbf{b}||\mathbf{d}|} = \cos 60^\circ = \frac{1}{2}$$



view from directly above
angles are with l_1, not as seen here

Squaring to remove the modulus and the square roots:

$$(q+r)^2 = \tfrac{1}{4}(p^2+q^2+r^2)(2) \;\Rightarrow\; 2(q+r)^2 = p^2+q^2+r^2$$

Now use $q = -2p - 2r$, so that $q + r = -(2p + r)$:

$$2(2p+r)^2 = p^2 + 4(p+r)^2 + r^2$$

$$8p^2 + 8pr + 2r^2 = 5p^2 + 8pr + 5r^2$$

$$p^2 = r^2 \;\Rightarrow\; r = p \;\text{ or }\; r = -p$$

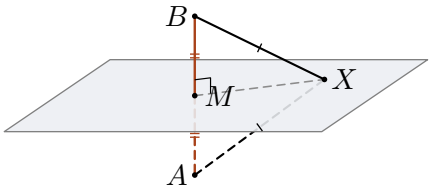
If $r = p$ then $q = -4p$, and if $r = -p$ then $q = 0$. Taking $p = 1$ in each case:

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-4\\1\end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

(f) 45° is the smallest angle that l_1 makes with any line in Π through P. Since 30° is smaller than that, no such line exists.

Question 11

(a) Let X be the point (x, y, z). Work with squared distances to avoid square roots. $AX^2 = BX^2$ says



$$(x-3)^2 + (y-1)^2 + (z-2)^2$$
$$= (x+1)^2 + (y-3)^2 + (z-6)^2$$

When both sides are expanded, each has $x^2 + y^2 + z^2$, and these cancel. What is left is

$$-6x - 2y - 4z + 14 = 2x - 6y - 12z + 46$$
$$-8x + 4y + 8z = 32$$
$$\boxed{2x - y - 2z = -8}$$

(b) The midpoint of AB is $M(1, 2, 4)$. It is on Π_1 because

$$2(1) - 2 - 2(4) = -8$$

AB is perpendicular to Π_1 because $\overrightarrow{AB}$ is a multiple of the normal of Π_1:

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

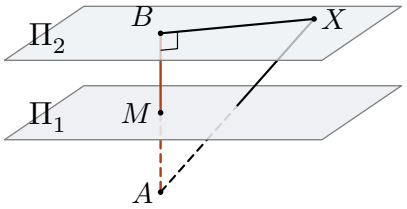
(c) Using the expansions from **(a)**:

$$AX^2 - BX^2 = -8x + 4y + 8z - 32$$

Setting this equal to 36:

$$-8x + 4y + 8z = 68 \;\Rightarrow\; \boxed{2x - y - 2z = -17}$$

(d) Π_2 has the same normal as Π_1, so it is also perpendicular to AB.



B lies on Π_2, because

$$2(-1) - 3 - 2(6) = -17$$

So for a point X on Π_2, BX lies in Π_2 and AB is normal to Π_2. So BX is perpendicular to AB, and the angle ABX is a right angle.

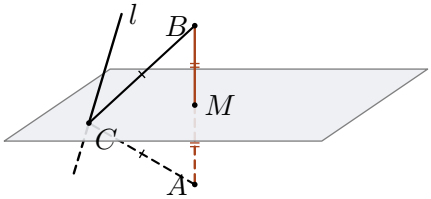
By Pythagoras in triangle ABX:

$$AX^2 = AB^2 + BX^2$$

$$AX^2 - BX^2 = AB^2 = 16 + 4 + 16 = 36$$

So 36 is AB^2: the value that makes the plane pass through B.

(e) $AC = BC$ means that C is on Π_1. It is also on l. So C is where l meets Π_1:



$$2(1) - (1 + t) - 2(-2t) = -8$$

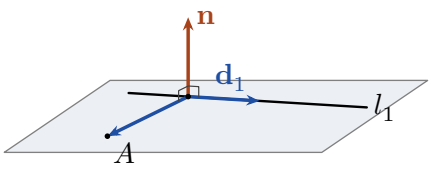
$$1 + 3t = -8 \;\Rightarrow\; t = -3$$

$$C(1, -2, 6)$$

$$AC = \sqrt{(-2)^2 + (-3)^2 + 4^2} = \sqrt{29}$$

Question 12

(a) Two directions in Π_1 are the direction $\mathbf{d}_1$ of l_1, and the vector to A from the point $(2, 2, -3)$ on l_1. These are the two blue arrows:



$$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

Let the normal have components a, b, c. It is perpendicular to both:

$$a + 2c = 0$$

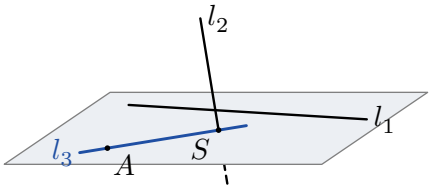
$$-b + 3c = 0$$

Taking $c = 1$ gives $a = -2$ and $b = 3$, so the plane is $-2x + 3y + z = k$.

A is on the plane, so $k = -4 + 3 + 0 = -1$.

$$2x - 3y - z = 1$$

(b) Any line through A that meets l_1 joins two points of Π_1, so it stays inside Π_1. So l_3 lies in Π_1.



l_3 also meets l_2. The point where it does so is on l_2 and in Π_1, and there is only one such point: S, where l_2 crosses Π_1.

$$2(0) - 3(-3 + \mu) - \mu = 1$$

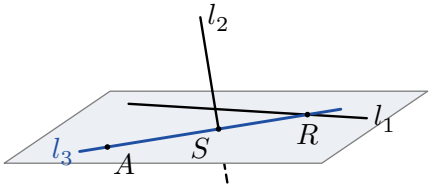
$$9 - 4\mu = 1 \;\Rightarrow\; \mu = 2$$

So S is $(0, -1, 2)$, and l_3 can only be the line through A and S. (Part **(c)** confirms that this line does meet l_1.)

$\overrightarrow{SA}$ has components 2, 2, -2, or more simply 1, 1, -1:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(c) l_3 meets l_2 at $S(0, -1, 2)$.



For l_1, set a general point of l_3 equal to a general point of l_1:

$$(2 + s,\ 1 + s,\ -s) = (2 + \lambda,\ 2,\ -3 + 2\lambda)$$

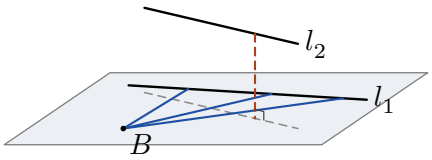
From y: $1 + s = 2$, so $s = 1$.

From x: $2 + s = 2 + \lambda$, so $\lambda = 1$.

Check in z: $-s = -1$ and $-3 + 2\lambda = -1$.

All three agree, so l_3 meets l_1 at $R(3, 2, -1)$.

(d) Any line through B that meets l_1 lies in the plane containing B and l_1 (the blue lines). So find that plane.



Two directions in it are the direction of l_1 and the vector from $(2, 2, -3)$ to B:

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

For the normal, with components a, b, c:

$$a + 2c = 0$$

$$2a - b + 3c = 0$$

Taking $c = 1$ gives $a = -2$ and $b = -1$. Multiplying the normal by -1 for convenience, the plane is $2x + y - z = k$, and B gives $k = 9$.

The direction of l_2 is perpendicular to this normal:

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

So l_2 is parallel to the plane. Its point $(0, -3, 0)$ gives $2(0) - 3 - 0 = -3$, not 9, so l_2 is not in the plane and never meets it.

Every line through B that meets l_1 stays in this plane, and l_2 never meets the plane. So no line through B meets both.